ALGEBRAIC NUMBER THEORY MIDTERM EXAM

Attempt all questions. Total : 100 marks

(1) Let m be a square free integer, let $K = \mathbb{Q}[\sqrt{m}]$. Prove that the ring of integers $R = \mathbb{A} \cap K$ of K has an integral basis $\{1, \sqrt{m}\}$ if $m \equiv 2$ or $3 \pmod 4$, and $\{1, \frac{1+\sqrt{m}}{2}$ if $m \equiv 1 \pmod 4$. Compute the discriminant in each case. [10+10=20 marks]

(2) Prove that $2.3 = (1+\sqrt{-5}).(1-\sqrt{-5})$ gives an example of non-unique factorization in $\mathbb{Z}[\sqrt{-5}]$. Show how both sides factor into products of prime ideals, and check the details [10+10=20 marks]

(3) Let $f(x) = x^3 + ax + b$ with $a, b \in \mathbb{Z}$. Assume that f is irreducible over $\mathbb{Q}$. Let α be a root of f.
   (a) Show that $\operatorname{disc}(\alpha) = -(4a^3 + 27b^2)$. (Hint: Use the fact that $\operatorname{disc}(\alpha) = -N_{\mathbb{Q}}^{\mathbb{Q}[\alpha]}(f'(\alpha))$, in this case).[10 marks]
   (b) Suppose $a = b = -1$, then show that the number ring $\mathbb{A} \cap \mathbb{Q}[\alpha] = \mathbb{Z}[\alpha]$ (you may quote a relevant result done in class).[10 marks]
   (c) Show that $23\mathbb{Z}[\alpha] = (23, \alpha - 10)^2(23, \alpha - 3)$. [10 marks]
   (d) Show that the ideals $(23, \alpha - 10)$ and $(23, \alpha - 3)$ are pairwise co-maximal prime ideals. [10 marks]

(4) Let K and L be number fields, $K \subset L$, $R = \mathbb{A} \cap K$, and $S = \mathbb{A} \cap L$ be the respective number rings. Let I and J be ideals of R, suppose IS divides JS (in S). Show that I divides J in R (you may assume that every nonzero ideal factorizes into prime ideals uniquely in a Dedekind domain). Show that for each ideal I in R, one has $I = IS \cap R$. [10+10 = 20 marks]